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THE REVERSIBILITY OF MECHANICAL SYSTEMS*

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Mechanical systems under the action of positional forces or a combination of positional forces and forces of the type of fourth-order forms with respect to the velocities (non-holonomic Chaplygin systems) as well as non-holonomic systems where there are no dissipative forces, belong to systems with a linear automorphism of a special type. In the case of such systems, stability of the equilibrium positions is only possible in the critical case of some zero and purely imaginary roots; asymptotic stability is impossible when there are zero roots. In the non-resonant case there is formal stability and a family of periodic motions exists similar to a Lyapunov family in the case of Hamiltonian systems and a family of conditionally periodic motions with a set of frequencies which are proportional to the frequencies of a linear system. The problem of the stability in the case of the lower (third and fourth) order resonances is solved. Examples are considered.

1. ***Reversible systems with a linear automorphism.*** The system

$$\mathbf{x}_*^{\cdot} = \mathbf{A}_*\mathbf{x}_* + \mathbf{X}_*(\mathbf{x}_*), \quad \mathbf{X}_*(0) = 0; \quad \mathbf{x}_*, \mathbf{X}_* \in \mathbf{R}^s \tag{1.1}$$

($\mathbf{A}_*$ is a constant $(s \times s)$ matrix) with a linear automorphism $\mathbf{M}$:

$$\mathbf{MA}_* = -\mathbf{A}_*\mathbf{M}, \quad \mathbf{MX}_*(\mathbf{x}_*) = -\mathbf{X}_*(\mathbf{Mx}_*), \quad \mathbf{M}^2 = \mathbf{E}$$

($\mathbf{E}$ is the unit matrix) is considered. This system is a special case of a system which is reversible in the sense of Birkhoff /1/ and has been investigated by Moser /2/.

The characteristic equation of the linear approximation $\det \| \mathbf{A}_* - \lambda\mathbf{E} \| = 0$ contains a root $-\lambda$ together with the root λ.

Actually,

$$\mathbf{M}(\mathbf{A}_* - \lambda\mathbf{E}) = -(\mathbf{A}_* + \lambda\mathbf{E})\mathbf{M}, \quad \det\|\mathbf{A}_* + \lambda\mathbf{E}\| = \det\|\mathbf{A}_* - \lambda\mathbf{E}\|$$

Such a system can therefore only be stable in the critical case of zero and purely imaginary roots. Let there be m zero and n pairs of purely imaginary roots and, moreover, let all the elementary divisors be simple. Then, using the non-degenerate transformation $x_* = \mathbf{P}y_*$, we reduce system (1.1) to the form

$$\xi^{\cdot} = \Xi(\xi, \eta, \bar{\eta}), \quad \eta^{\cdot} = \Lambda\eta + \Phi(\xi, \eta, \bar{\eta}), \quad \bar{\eta}^{\cdot} = -\Lambda\bar{\eta} + \bar{\Phi}(\xi, \eta, \bar{\eta}) \tag{1.2}$$
$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

where ξ is a real m-vector and η and $\bar{\eta}$ are complex-conjugate n-vectors. System (1.2) has the linear automorphism $\mathbf{M}_* = \mathbf{P}^{-1}\mathbf{MP}$, $\mathbf{M}_*^2 = \mathbf{E}$ /3/. The linear part of system (1.2) is invariant under the substitutions: 1) $t \to -t$, $\xi \to \xi$, $\eta \to \bar{\eta}$, $\bar{\eta} \to \eta$ and 2) $t \to -t$, $\xi_1 \to \xi_1, \ldots, \xi_k \to \xi_k$, $\xi_{k+1} \to -\xi_{k+1}, \ldots, \xi_m \to -\xi_m$, $\eta \to \bar{\eta}$, $\bar{\eta} \to \eta$. Hence, the automorphism $\mathbf{M}_*$ may be of the form

$$N = \begin{Vmatrix} \mathbf{E}_m & 0 & 0 \\ 0 & 0 & \mathbf{E}_n \\ 0 & \mathbf{E}_n & 0 \end{Vmatrix} \quad \text{and} \quad L = \begin{Vmatrix} \mathbf{E}_k & 0 & 0 & 0 \\ 0 & -\mathbf{E}_{m-k} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{E}_n \\ 0 & 0 & \mathbf{E}_n & 0 \end{Vmatrix}$$

where $\mathbf{E}_j$ is a unit j-matrix.

Example 1 /2/. The system

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$$\mathbf{y}^{\cdot} = \mathbf{Y}(\mathbf{y}, \mathbf{z}), \quad \mathbf{z}^{\cdot} = \mathbf{Z}(\mathbf{y}, \mathbf{z}); \quad \mathbf{y} \in \mathbf{R}^{l}, \quad \mathbf{z} \in \mathbf{R}^{n} \tag{1.3}$$

$$\mathbf{Y}(\mathbf{y}, -\mathbf{z}) = -\mathbf{Y}(\mathbf{y}, \mathbf{z}), \quad \mathbf{Z}(\mathbf{y}, -\mathbf{z}) = \mathbf{Z}(\mathbf{y}, \mathbf{z}); \quad M = \begin{Vmatrix} -\mathbf{E}_l & 0 \\ 0 & \mathbf{E}_n \end{Vmatrix}$$

with a linear automorphism $\mathbf{M}$ is encountered in a number of mechanical problems /4-6/. The linear approximation necessarily has the form

$$\mathbf{y}^{\cdot} = \mathbf{B}\mathbf{z}, \quad \mathbf{z}^{\cdot} = \mathbf{C}\mathbf{y}$$

Let $\operatorname{rank} \mathbf{B} = l_1$ and $\operatorname{rank} \mathbf{C} = n_1$. This system then reduces to the form

$$\mathbf{y}_1^{\cdot} = 0, \quad \mathbf{y}_2^{\cdot} = \mathbf{B}_2\mathbf{z}_2^*, \quad \mathbf{z}_1^{\cdot} = 0, \quad \mathbf{z}_2^{\cdot} = \mathbf{C}_2\mathbf{y}_2^*; \quad \mathbf{y}_2, \mathbf{z}_2^* \in \mathbf{R}^{l_1}, \quad \mathbf{z}_2, \mathbf{y}_2^* \in \mathbf{R}^{n_1}$$

$$\operatorname{rank} \mathbf{B}_2 = l_1, \ \operatorname{rank} \mathbf{C}_2 = n_1$$

It is therefore mandatory that the characteristic equation possesses $m = l + n - 2n^*$, $n^* = \min(l_1, n_1)$ zero roots with $l + n - (l_1 + n_1)$ groups of solutions. If $l_1 = n_1 = n^*$, then all the elementary divisors are simple and the system can only be stable in the critical case of m zero and n^* pairs of purely imaginary roots. When $n^* < n$, there is an automorphism L and, when $n^* = n$, there is an automorphism N. In the case when $l < n$, we have $l_1 < n$ and only the automorphism L is possible.

Let us now consider system (1.2) with an automorphism N (system N) in greater detail. The relationships

$$\Xi(\xi, \eta, \bar{\eta}) = -\Xi(\xi, \bar{\eta}, \eta), \quad \Phi(\xi, \eta, \bar{\eta}) = -\bar{\Phi}(\xi, \bar{\eta}, \eta), \tag{1.4}$$

$$\bar{\Phi}(\xi, \eta, \bar{\eta}) = -\bar{\Phi}(\xi, \bar{\eta}, \eta)$$

must be satisfied in the case of this system which mean that the function Ξ does not contain terms which are free of η and $\bar{\eta}$ and depend solely on ξ. the essentially specific critical case of m zero roots occurs. On the other hand, conditions (1.4) guarantee that the coefficients in the expansion of the functions Ξ, Φ and $\bar{\Phi}$ are purely imaginary. These conclusions lead to a number of remarkable properties of N systems.

1°. Asymptotic stability is impossible in the case of a reversible system N.

Actually, a family of steady-state motions $\xi = \mathbf{c}$, $\eta = \bar{\eta} = 0$ ($\mathbf{c}$ is a constant m-vector) exists in the case of such a system.

2°. A system N is formally stable in the non-resonant case of m zero and n pairs of purely imaginary roots.

This and also the next two properties are derived from the results /3, 7/* (*Also, see A.D. Bryuno, Sets of Analyticity of a Normalizing Transformation, Preprints 97 and 98, Inst. Prikl. Matem., Akad. Nauk SSSR, Moscow, 1974.) of an investigation into the normal forms when these results are applied to systems N. The automorphism of N is retained in the normal form /3/. Here, it is necessary to take the normalizing transformation with real coefficients. The normal form is then of the type (in writing this, we shall make use of the same variables as in (1.2)):

$$\xi^{\cdot} = 0, \ \eta_s^{\cdot} = i\eta_s\Psi_s(\rho, \xi), \ \bar{\eta}_s^{\cdot} = -i\bar{\eta}_s\Psi_s(\rho, \xi) \tag{1.5}$$

$$\Psi_s(\rho, \xi) = \omega_s + \sum_{j=1}^{n} A_{sj}\rho_j + \sum_{j=1}^{m} C_{sj}\xi_j + \sum_{j,k=1}^{m} D_{sjk}\xi_j\xi_k + \ldots$$

$$\rho = (\rho_1, \ldots, \rho_n); \quad \rho_s = \eta_s\bar{\eta}_s, \ \omega_s = |\lambda_s| \quad (s = 1, \ldots, n)$$

where $\Psi_s(\rho, \xi)$ are formal series in ρ and ξ with real coefficients. Property 2° therefore follows from the fixed-sign integral $\xi^2 + \eta\bar{\eta} = \text{const}$ of system (1.5).

According to /7/ (also, see the footnote), the given version (1.5) of the normal form guarantees the analyticity of the set $\mathbf{A}^* = A_1^* \cup A_2^* \cup \ldots \cup A_n^*$:

$$A_s^* = \{\rho, \xi : \rho_j = 0; \ j \neq s; \ \xi = 0\} \ (s = 1, \ldots, n) \tag{1.6}$$

The set A^* consists of n one-parameter families of periodic motions which are analogous to the Lyapunov families /8/ in the case of Hamiltonian systems.

3°. In the non-resonant case of m zero and n pairs imaginary roots, system N has a single-parameter family (a is the parameter) of randomly periodic motions

$$\mathbf{A}^0 = \{\rho, \xi : \xi = 0; \ \Psi_j = \omega_j a \ (j = 1, \ldots, n)\}$$

with a set of frequencies $(\omega_1 a, \ldots, \omega_n a)$, if ω satisfies the condition

$$|\langle \mathbf{q}, \omega \rangle| > \mu |\mathbf{q}|^{-\nu} \tag{1.7}$$

for all integer vectors $\mathbf{q}$ with $\langle \mathbf{q}, \boldsymbol{\omega}\rangle \neq 0$ (μ and ν are certain positive constants and $\det \mathbf{A} \neq 0$, $\mathbf{A} = \| A_{sj} \|$.

This follows from the identity of the set $\mathbf{A}^0$ to a certain analytic set $\mathbf{B}$ /7/. The proof of the nilpotency of the corresponding matrix $\mathbf{B}$ can be obtained by a word-by-word repetition of the arguments presented in the case of a Hamiltonian system which has also been noted by Bryuno (also, see the previous footnote).

2. Stability when there are resonances. It follows from property 2° that a problem on Lyapunov stability can only be solved in the resonant case on the basis of an analysis of the finite-order terms. The lower (third and fourth) order resonances are the most important in this respect. The resonance problem in the special case when $m = 0$ has been solved in /9/.

Initially, we note an important fact. Let system (1.2) be redued to the normal form up to terms of the K-th order and let resonances occur in the system. The lowest order of these resonances is equal to K^+. The function Ξ then begins with terms of not lower than the K-th order with respect to ξ, η and $\bar{\eta}$. Hence, if there is a third-order resonance in the system, then the model system, which is obtained from the normal form by discarding terms of higher than the second order, has a solution in which $\xi = 0$ and the variables $\eta, \bar{\eta}$ are described by a subsystem which is identical to the case when $m = 0$. This means that the conclusions drawn in /9/ concerning instability remain true. Moreover, the conclusions regarding the stability of the model system are also unchanged as, when $\xi = \mathrm{const}$, the equations for the polar radii $r_s = \eta_s \bar{\eta}_s$ are unchanged. Hence, a third-order resonance is solved by Theorem 1 from /9/.

Let us now consider the fourth-order resonances

$$\sum_{j=1}^{\mu} p_j \lambda_j = 0, \quad p_j > 0, \sum_{j=1}^{\mu} p_j = 4, \quad \mu \leqslant 4, \quad \mu \leqslant n$$

The model system, containing terms of the third-order inclusive, has the form

$$\xi_\gamma^{\cdot} = 0 \quad (\gamma = 1, \ldots, m) \tag{2.1}$$

$$\eta_\alpha^{\cdot} = \lambda_\alpha \eta_\alpha + i\eta_\alpha \Big(\sum_{\gamma=1}^{m} C_{\alpha\gamma} \xi_\gamma + \sum_{j=1}^{n} A_{\alpha j} r_j + \sum_{s,j=1}^{m} D_{\alpha sj} \xi_s \xi_j \Big) + iB_\alpha \prod_{j=1}^{\mu} \bar{\eta}_j^{p_j - \delta_{\alpha j}}$$

$$\eta_\beta^{\cdot} = \lambda_\beta \eta_\beta + i\eta_\beta \Big(\sum_{\gamma=1}^{m} C_{\beta\gamma} \xi_\gamma + \sum_{j=1}^{n} A_{\beta j} r_j + \sum_{s,j=1}^{m} D_{\beta sj} \xi_s \xi_j \Big)$$

$$(\alpha = 1, \ldots, \mu; \ \beta = \mu + 1, \ldots, n)$$

Here $A_{\alpha j}, A_{\beta j}, B_\alpha, C_{\alpha\gamma}, C_{\beta\gamma}, D_{\alpha sj}$ and $D_{\beta sj}$ are real constants and the complex-conjugate group of equations has been omitted.

Let us investigate the non-degenerate case when none of the coefficients B_α vanishes. In polar coordinates

$$\eta_s = \sqrt{r_s} \exp(i\theta_s), \ \bar{\eta}_s = \sqrt{r_s} \exp(-i\theta_s) \ (s = 1, \ldots, n)$$

system (2.1) takes the form

$$\xi_\gamma^{\cdot} = 0 \quad (\gamma = 1, \ldots, m) \tag{2.2}$$

$$r_\alpha^{\cdot} = 2B_\alpha \sin\theta \prod_{j=1}^{\mu} r_j^{p_j/2}; \quad r_\beta^{\cdot} = 0 \quad (\alpha = 1, \ldots, \mu; \ \beta = \mu + 1, \ldots, n)$$

$$\theta^{\cdot} = \varepsilon(\xi) + \prod_{j=1}^{n} A_j r_j + \sum_{j=1}^{\mu} p_j B_j \prod_{k=1}^{\mu} r_k^{p_k/2 - \delta_{jk}} \cos\theta$$

$$\theta_\beta^{\cdot} = \omega_\beta + \sum_{j=1}^{m} C_{\beta j} \xi_j + \sum_{j=1}^{n} A_{\beta j} r_j + \sum_{j,k=1}^{m} D_{\beta jk} \xi_j \xi_k$$

$$A_j = \sum_{\alpha=1}^{\mu} p_\alpha A_{\alpha j}, \quad \theta = \sum_{\alpha=1}^{\mu} p_\alpha \theta_\alpha, \quad \varepsilon(\xi) = \sum_{\alpha=1}^{\mu} p_\alpha \Big(\sum_{j=1}^{m} C_{\alpha j} \xi_j + \sum_{j,k=1}^{m} D_{\alpha jk} \xi_j \xi_k \Big)$$

$$\omega_\beta = -i\lambda_\beta$$

It can be shown by direct substitution that system (2.2) has the following first integrals:

$$W_{s0} \equiv \xi_s = h_s^0 \quad (s = 1, \ldots, m) \tag{2.3}$$

$$W_\alpha \equiv B_\alpha r_1 - B_1 r_\alpha = h_\alpha \quad (\alpha = 2, \ldots, \mu)$$

$$W_\beta \equiv r_\beta = h_\beta \quad (\beta = \mu + 1, \ldots, n)$$

$$W \equiv \Big(\sum_{\alpha=1}^{\mu} A_\alpha \prod_{j=1}^{\mu} B_j^{1-\delta_{\alpha j}}\Big) r_\alpha^2 + 4\prod_{j=1}^{\mu} B_j r_j^{p_j/2}\cos\theta + \frac{2}{\mu}\Big(\sum_{\alpha=1}^{\mu} r_\alpha \prod_{j=1}^{\mu} \beta_j^{1-\delta_{\alpha j}}\Big)\Big(\varepsilon(\xi) + \sum_{\beta=\mu+1}^{n} A_l r_\beta\Big) = h$$

where h_s^0, h_ν $(s = 1, \ldots, m;\ \nu = 2, \ldots, n)$ and h are arbitrary constants. Consequently, if there is a change in sign in the sequence of coefficients $B_1, \ldots, B_\mu$, then it is possible to make up a fixed-sign integral which is linear with respect to $\xi_1^2, \ldots, \xi_m^2$ and $r_1, \ldots, r_n$ from the first three groups of integrals (2.3) which proves the stability of the model system (2.1).

Let all the B_α be of the same sign. From the integrals of (2.3), we compose the function

$$V(\mathbf{r}, \xi, \theta) = W^2 + \sum_{\nu=2}^{n} W_\nu^2 + \sum_{s=1}^{m} W_{s0}^2$$

It is obvious that the function V will be positive-definite with respect to $\xi_1, \ldots, \xi_m$, $r_1, \ldots, r_n$ if we have $W \neq 0$ on the manifold

$$W_\nu = 0\ (\nu = 2, \ldots, n),\ W_{s0} = 0\ (s = 1, \ldots, m) \tag{2.4}$$

In (2.4), the function W has the form

$$W_* = B_1^{-2}\prod_{j=1}^{\mu} B_j(\Sigma + 4\Pi\cos\theta)\, r_1^2;\quad \Sigma = \sum_{\alpha=1}^{\mu} A_\alpha B_\alpha,\quad \Pi = \prod_{j=1}^{\mu} |B_j|^{p_j/2}$$

and does not vanish if

$$|\Sigma| > 4\Pi \tag{2.5}$$

Hence, if all the B_α are of the same sign, then, when condition (2.5) is satisfied, V is a Lyapunov function in the case of (2.3) which satisfies the stability theorem.

Now let the sign of the inequality in (2.5) change into the opposite sign and all the B_α be of the same sign. The model system then has an increasing solution in the form of a ray

$$r_\alpha = \gamma_\alpha r,\ r_\beta = 0,\ \xi_s = 0,\ r^{\boldsymbol\cdot} = \gamma r^2,\ \theta \doteq \theta_0;\ \gamma,\ \gamma_\alpha > 0$$
$$(\alpha = 1, \ldots, \mu;\ \beta = \mu + 1, \ldots, n;\ s = 1, \ldots, m)$$

Actually, by substituting this solution into (2.2), we get

$$\mathbf{r}^{\boldsymbol\cdot} = 2r^2\Big(\frac{B_\alpha}{\gamma_\alpha}\prod_{j=1}^{\mu}\gamma_j^{p_j/2}\Big)\sin\theta,\quad \theta^{\boldsymbol\cdot} = \Big[\sum_{\alpha=1}^{\mu} A_\alpha\gamma_\alpha + 4B_1\prod_{j=1}^{\mu}\gamma_j^{p_j/2}\cos\theta\Big] r$$

whence

$$\gamma_1 = 1,\ \gamma_2 = B_2/B_1, \ldots, \gamma_\mu = B_\mu/B_1;\ 4\,|\cos\theta_0|\,\Pi = |\Sigma|,\ B_1\sin\theta_0 > 0$$

Consequently, the model system is unstable in this case. The instability of the overall system is proved in the same way as in /9/.

Theorem 1. If $B_\alpha \neq 0\ (\alpha = 1, \ldots, \mu)$ then, in order for the model system (2.1) to be stable, it is necessary and sufficient that one of the following conditions should be satisfied: a) a pair of coefficients B_j and B_k of opposite sign exist, b) all the coefficient B_α are of the same sign and condition (2.5) is satisfied.

If, however, all the B_α are of the same sign and the sign in inequality (2.5) is changed, then the zeroth solution of the initial system is unstable in the Lyapunov sense.

3. ***The reversibility of mechanical systems.*** A Hamiltonian system is an important example of a mechanical system which is reversible in the Birkhoff sense /1/. Below, we consider examples of mechanical systems with an automorphism N.

1°. A mechanical system under the action of positional forces. A holonomic mechanical system with n degrees of freedom which is constrained by stationary geometric links and subjected to the action of positional forces is described by the equations:

$$\frac{\partial}{dt}\frac{\partial T}{\partial q_s^{\cdot}}-\frac{\partial T}{\partial q_s}=Q_s\ (s=1,\ldots,n),\quad 2T=\sum_{i,j=1}^{n}a_{ij}(\mathbf{q})\,q_i^{\cdot}q_j^{\cdot} \tag{3.1}$$

where T is the kinetic energy and the generalized forces Q_s are solely dependent on the coordinates $\mathbf{q}$. System (3.1) is invariant under the linear substitution M^*: $t\to -t$, $\mathbf{q}\to\mathbf{q}$, $\mathbf{q}^{\cdot}\to -\mathbf{q}^{\cdot}$. In the neighbourhood of the equilibrium position when the characteristic equation has purely imaginary roots, the system of equations of the perturbed motion possesses a linear automorphism $N\ (m=0)$. The corresponding matrix $\mathbf{P}$ of the linear transformation has been obtained in /9/.

2°. Non-holonomic Chaplygin system. In this case the equations of motion can also be taken in the form of (3.1) /10/ only here the generalized forces (when there is no dissipation) have the form

$$Q_s=Q_s^{\circ}(\mathbf{q})+\sum_{i,j=1}^{n}f_{sij}(\mathbf{q})\,q_i^{\cdot}q_j^{\cdot}$$

The system also has a linear automorphism M^* and since the supplementary forces are quadratic with respect to the velocities, the linear transformation matrix $\mathbf{P}$ is the same as in /9/.

3°. A non-holonomic system when there are no dissipative forces. The equations of motion of a non-holonomic system can be taken /11/ in the Voronets form

$$\frac{\partial}{dt}\frac{\partial\Theta}{\partial q_r^{\cdot}}=\frac{\partial(\Theta+U)}{\partial q_r}+\sum_{\varkappa=l+1}^{n}\frac{\partial(\Theta+U)}{\partial q_\varkappa}b_{\varkappa r}(\mathbf{q})+\sum_{\varkappa=l+1}^{n}\Theta_\varkappa\sum_{s=1}^{l}\nu_{\varkappa rs}q_s^{\cdot}\ \ (r=1,\ldots,l) \tag{3.2}$$

$$2\Theta=\sum_{r,s=1}^{l}\tau_{rs}(\mathbf{q})\,q_r^{\cdot}q_s^{\cdot},\quad \Theta_\varkappa=\sum_{p=1}^{l}\theta_{\varkappa p}(\mathbf{q})\,q_p^{\cdot}$$

$$\nu_{\varkappa rs}=\frac{\partial b_{\varkappa r}}{\partial q_s}-\frac{\partial b_{\varkappa s}}{\partial q_r}-\sum_{\varkappa'=l+1}^{n}\left(b_{\varkappa' s}\frac{\partial b_{\varkappa r}}{\partial q_{\varkappa'}}-b_{\varkappa' r}\frac{\partial b_{\varkappa s}}{\partial q_{\varkappa'}}\right)$$

(U is a force function) and the equations of the connections are represented in the form

$$q_\varkappa^{\cdot}=\sum_{r=1}^{l}b_{\varkappa r}(\mathbf{q})\,q_r^{\cdot}\quad(\varkappa=l+1,\ldots,n) \tag{3.3}$$

It can be seen that system (3.2), (3.3) also has a linear automorphism M^*. In the neighbourhood of the investigated equilibrium position

$$q_r=q_{r0},\ q_r^{\cdot}=0\ (r=1,\ \ldots,\ l),\ q_\varkappa=q_{\varkappa 0}\ (\varkappa=l+1,\ \ldots,\ n)$$

the equations of the perturbed motion have the form /11/

$$\sum_{s=1}^{l}a_{rs}x_s^{\cdot\cdot}+\sum_{s=1}^{l}(c_{rs}-e_{rs})\,x_s+\sum_{\varkappa=l+1}^{n}p_{r\varkappa}x_\varkappa=X_r \tag{3.4}$$
$$x_\varkappa^{\cdot}=X_\varkappa\ \ (r=1,\ \ldots,\ l;\ \varkappa=l+1,\ \ldots,\ n)$$

(a_{rs}, c_{rs}, e_{rs} and $p_{r\varkappa}$ and the expansions of the functions X contain terms of not lower than the second order in $x_1,\ldots,x_n,x_1^{\cdot},\ldots,x_l^{\cdot}$) if one puts

$$q_r=q_{r0}+x_r\,(r=1,\ldots,l);\quad q_\varkappa=q_{\varkappa 0}+x_\varkappa+\sum_{r=1}^{l}b_{\varkappa r}(\mathbf{q}_0)\,x_r\quad(\varkappa=l+1,\ldots,n) \tag{3.5}$$

in the perturbed motion.
Here, the functions $X_\varkappa$ are calculated using the formulae

$$X_\varkappa(x_1,\ldots,x_n,x_1^{\cdot},\ldots,x_l^{\cdot})=\sum_{r=1}^{l}[b_{\varkappa r}(\mathbf{q})-b_{\varkappa r}(\mathbf{q}_0)]\,|_{(3.5)}\,x_r^{\cdot} \tag{3.6}$$

The characteristic equation of system (3.4) has at least $n-l$ zero roots and, moreover, a group of equations in $x_\varkappa$ corresponds to the $n-l$ zero roots. In the linear approximation, this group of equations reduces to the form (1.2) and the linear part of system (3.4) is invariant under the replacement $t\to -t$, $\mathbf{x}\to\mathbf{x}$, $\mathbf{x}^{\cdot}\to -\mathbf{x}^{\cdot}$. Consequently, when the number of zero roots is equal to the number of non-holonomic links and the remaining roots are purely imaginary, there is an automorphism N.

Theorem 2. When there are no dissipative forces, the equlibrium position of a non-holonomic system is formally stable if the number of zero roots of the characteristic equation when the remaining roots are purely imaginary, $\pm i\omega_s$, is equal to the number of non-holonomic links and the numbers ω_s are linearly independent over rational numbers.

Theorem 3. Under the conditions of Theorem 2 there exists l single-parameter families of periodic motions adjacent to the equlibrium position which is being considered.

This conclusion follows from property 3° of Sect.1.

System (3.2), (3.3) has an energy integral $\Theta - U = h - \text{const}$. If equations (3.4) are now reduced to the form of (1.5), then the function $\Theta - U$ reduces to the form

$$\Theta - U = \sum_{j=1}^{n-l} \alpha_j \xi_j + \sum_{k=1}^{l} \beta_k \rho_k + \ldots$$

where α_j and β_k are certain constants. The system of equations

$$\xi_j = 0 \quad (j = 1, \ldots, n - l)$$

$$\sum_{j=1}^{n-l} C_{sj}\xi_j + \sum_{j=1}^{l} A_{sj}\rho_j + \sum_{j,k=1}^{n-l} D_{sjk}\xi_i\xi_k + \ldots = \omega_s (a - 1) \quad (s = 1, \ldots, l)$$

$$\sum_{j=1}^{n-l} \alpha_j \xi_j + \sum_{k=1}^{l} \beta_k \rho_k + \ldots = h$$

therefore has a unique solution which depends on h if

$$\det \begin{Vmatrix} \mathbf{A} & \Omega \\ \boldsymbol{\beta} & 0 \end{Vmatrix} \neq 0; \quad \Omega = \operatorname{col}(\omega_1, \ldots, \omega_l), \quad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_l) \tag{3.7}$$

The following theorem therefore follows from property 4° of Sect.1.

Theorem 4. Under the conditons of Theorem 2, in the neighbourhood of the equilibrium position which is being considered there exists a single-parameter family (h is the parameter) of random-periodic motions with a set of frequencies $(\omega_1 a(h), \ldots, \omega_l a(h))$ adjacent to this equilibrium, if conditions (1.7) and (3.7) are additionally satisfied.

Remark. If all the constants $e_{rs} = 0$ in Eqs.(3.4), then the numbers β_s are identical to ω_s.

The question of the reversibility of a mechanical system as a function of the forces which are acting is of interest. It is clear that dissipative forces lead to the non-reversibility of the system. In the general case, a system under the action of positional and gyroscopic forces is also irreversible. This can already be seen by considering the example of a mechanical system with two degrees of freedom.

Example 2. The system

$$q_1^{\cdot\cdot} = \alpha q_1 + \gamma q_2^{\cdot} + \varepsilon q_2, \quad q_2^{\cdot\cdot} = \beta q_2 - \gamma q_1^{\cdot} - \varepsilon q_1 \tag{3.8}$$

(α, β, γ and ε are constants) is not gyroscopically coupled and reversible: $t \to -t$, $q \to q$, $q^{\cdot} \to -q^{\cdot}$ and, when $\varepsilon = 0$, it finds itself solely under the action of potential and gyroscopic forces and is reversible: $t \to -t$, $q_1 \to q_1$, $q_1^{\cdot} \to -q_1^{\cdot}$, $q_2 \to -q_2$, $q_2^{\cdot} \to q_2^{\cdot}$. In the general case, when $\gamma\varepsilon \neq 0$, the characteristic equation

$$\lambda^4 + (\gamma^2 - \alpha - \beta)\lambda^2 + 2\gamma\varepsilon\lambda + \varepsilon^2 + \alpha\beta = 0$$

does not have a root $-\lambda$ together with the root λ and the system is irreversible.

4°. *A homogeneous ellipsoid on a rough plane.* A reversible system N arises not only in the investigation of a non-holonomic system in the neighbourhood of the equilibrium position. For example, in the problem which has been indicated when studying motions close to stationary rotations around the vertical, it is convenient to take the equations in the Appel form. Retaining the notation employed in /4/, we have

$$[A + m(y^2 + z^2)]\, p^{\cdot} - mxyq^{\cdot} - mxzr^{\cdot} = \tag{3.9}$$
$$(B - C)\, qr + m(x^{\cdot} - yr + zq)\langle \boldsymbol{\omega}, \mathbf{r}_\mu \rangle - mp \langle \mathbf{r}_\mu, \mathbf{r}_\mu^{\cdot} \rangle +$$
$$mga^2 (c^2 - b^2)\, \Delta^{-1} yz$$
$$x^{\cdot} = yr - zq + \frac{a^2 - c^2}{a^2c^2}(x^2 - a^2)\, zq + \frac{b^2 - a^2}{b^2a^2}(x^2 - a^2)\, yr + \frac{c^2 - b^2}{c^2b^2} xyzp$$
$$(pqr, \;\; xyz, \;\; ABC, \;\; abc)$$

where

$$\omega = (p,\ q,\ r),\ \mathbf{r}_\mu = (x,\ y,\ z),\ \mathbf{r}_\mu^\cdot = (x^\cdot,\ y^\cdot,\ z^\cdot)$$

and the coordinates are linked by the relationship.

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

This system of equations has the particular solution

$$p = q = 0,\ r = \omega = \text{const},\ x = y = 0,\ z = -c$$

which corresponds to the rotation of the ellipsoid about the vertical and represents a single-parameter family (ω is the parameter).

An analysis of the system shows the presence of a linear automorphism M: $t \to -t,\ (x, p, z, r) \to (x, p, z, r),\ (y, q) \to (-y, -q)$. The equations of the perturbed motion are derived from (3.9) by replacing r by $r + \omega$ and z by $z - c$. Consequently, these equations have the above-mentioned automorphism M and are a system of the form of (1.3). Here $n = 2,\ l = 2,\ n_1 = l_1 = 2$ and the matrices B_2 and C_2 have the form

$$B_2 = \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \frac{a^2}{b^2}\omega & \frac{a^2}{c} \end{Vmatrix}, \quad C_2 = \begin{Vmatrix} \beta_{11} & \beta_{12} \\ -\frac{b^2}{c}\omega & -\frac{b^2}{c} \end{Vmatrix}$$

$$\alpha_{11} = \omega\frac{6c^2 - 5a^2 - b^2}{b^2 + 6c^2}, \quad \alpha_{12} = 5\frac{c\omega^2(b^2 - a^2) + g(b^2 - c)}{b^2(b^2 + 6c^2)}$$

$$\beta_{11} = \omega\frac{a^2 + 5b^2 - 6c^2}{a^2 + 6c^2}, \quad \beta_{12} = 5\frac{c\omega^2(b^2 - a^2) + g(c^2 - a^2)}{a^2(a^2 + 6c^2)}$$

Hence, in the critical case of two zero and two pairs of purely imaginary roots, all the conclusion of Sect.1 hold and, in particular, there exist two families of periodic motions close to stationary rotations. We note that the stability when there are lower-order resonances in this problem has been investigated in /12, 13/.

5°. *A heavy solid with a fixed point.* The Euler-Poisson equations of this problem (/14/. p.177)

$$A\frac{dp}{dt} + (C - B)qr = P(\gamma_2 z_c - \gamma_3 y_c), \quad \frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3$$
$$(pqr,\ \gamma_1\gamma_2\gamma_3,\ ABC,\ x_c y_c z_c)$$

represent a reversible system with the linear automorphism:

$$t \to -t,\ (p,\ q,\ p) \to (-p,\ -q,\ -r),\ (\gamma_1,\ \gamma_2,\ \gamma_3) \to (\gamma_1,\ \gamma_2,\ \gamma_3).$$

6°. *An unbounded three-body problem.* The Routh-Lyapunov equations for this classical problem in celestial mechanics have the form (/15/, p.397)

$$\frac{d^2r_1}{dt^2} - r_1(\omega_2^2 + \omega_3^2) + f(m_0 + m_1)F(r_1) + fm_2[F(r_2)\cos\psi + F(\Delta)\cos\varphi_1] = 0$$
$$\frac{1}{r_1}\frac{d}{dt}(r_1^2\omega_3) + r_1\omega_2\omega_3 + fm_2[F(r_2)\sin\psi - F(\Delta)\sin\varphi_1] = 0$$
$$\frac{1}{r_1}\frac{d}{dt}(r_1^2\omega_2) - r_1\omega_1\omega_3 = 0$$
$$\Delta^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\psi, \quad \sin\varphi_i = \frac{r_i}{\Delta}\sin\psi \quad (i = 1, 2)$$
$$\omega_1^* = \omega_1\cos\psi + \omega_2\sin\psi, \quad \omega_2^* = -\omega_1\sin\psi + \omega_2\cos\psi, \quad \omega_3^* = \omega_3 + \psi^\cdot$$

(the second group of differential equations is obtained from the equations presented above by replacing $(r_1,\ \omega_1,\ \omega_2,\ \omega_3,\ \varphi_1$ and $\psi)$ by $(r_2,\ \omega_1^*,\ \omega_2^*,\ \omega_3^*,\ -\varphi_2,\ -\psi)$ and $(m_1,\ m_2)$ by $(m_2,\ m_1)$. $F(r)$ is a certain function of the reciprocal distance r). The system is reversible with the linear automorphism $t \to -t,\ (r_1,\ r_2,\ \psi) \to (r_1,\ r_2,\ \psi),\ (\omega_1,\ \omega_2,\ \omega_3) \to (-\omega_1,\ -\omega_2,\ -\omega_3)$.

4. *Example*. The problem of the relative equilibrium and regular precession of a geostationary satellite /5, 6/ may be cited as a further example of a complex multiparameter mechanical system which leads to the investigation of a system with a linear automorphism N. However, this problem is the subject of a separate treatment. Here, we shall confine ourselves to the treatment of an illustrative example.

In the case of a model of an elastic rod under the action of a servoforce /9/ we have a system under the action of positional forces. The linear problem has been analysed /9/, the transformation to the normal form has been obtained and the lower order resonances have been investigated. Retaining the notation employed in /9/, let us consider the non-resonant case. In this case, non-linear normalization is carried out in the complex-conjugate variables z and $\bar{z}$ which are related to the angles φ_1 and φ_2 of deviation of the rod from the equilibrium position and their velocities by the relationships

$$\varphi_1 = \frac{b_{12}}{2(\omega_1^2-\omega_2^2)}\left[\frac{z_1+\bar z_1}{\omega_1}-\frac{z_2+\bar z_2}{\omega_2}\right]+\ldots$$

$$\varphi_2 = \frac{1}{2(\omega_1^2-\omega_2^2)}\left[\frac{b_{22}+\omega_2^2}{\omega_1}(z_1+\bar z_1)-\frac{b_{22}+\omega_1^2}{\omega_2}(z_2+\bar z_2)\right]+\ldots$$

$$\varphi_1^{\cdot} = \frac{b_{12}i}{2(\omega_1^2-\omega_2^2)}[(z_1-\bar z_1)-(z_2-\bar z_2)]+\ldots$$

$$\varphi_2^{\cdot} = \frac{i}{2(\omega_1^2-\omega_2^2)}[(b_{22}+\omega_2^2)(z_1-\bar z_1)-(b_{22}+\omega_1^2)(z_2-\bar z_2)]+\ldots$$

The equations for the radii $r_1 = z_1\bar z_1$, $r_2 = z_2\bar z_2$ and the polar angles $\theta_1 = \arg z_1$, $\theta_2 = \arg z_2$ have the form

$$r_1^{\cdot} = 0,\ r_2^{\cdot} = 0,\ \theta_1^{\cdot} = \omega_1+\varepsilon_1(r_1, r_2),\ \theta_2^{\cdot} = \omega_2+\varepsilon_2(r_1, r_2)$$

Hence, two single-parameter families of periodic motions exist close to the equilibrium position. The first family is defined by the relationship

$$\varphi_1 = \frac{b_{12}\sqrt{r_{10}}}{\omega_1(\omega_1^2-\omega_2^2)}\cos[\omega_1+\varepsilon_1(r_{10}, 0)]t+\ldots$$

$$\varphi_2 = \frac{(b_{22}+\omega_2^2)\sqrt{r_{10}}}{\omega_1(\omega_1^2-\omega_2^2)}\cos[\omega_1+\varepsilon_1(r_{10}, 0)]t+\ldots$$

and the second corresponds to the replacement of ω_1 by ω_2, ω_2 by ω_1, $\varepsilon_1(r_{10}, 0)$ by $\varepsilon_2(0, r_{20})$ and r_{10} by r_{20} (r_{10} and r_{20} are the initial values of r_1 and r_2). These oscillations are close to the vibrations of the linear system and, moreover,

$$\varphi_1 = k\varphi_2+\ldots;\quad k = \frac{b_{12}}{b_{22}+\omega_2^2},\ \frac{b_{12}}{b_{22}+\omega_1^2} < 0$$

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